

RADIAL SCHRÖDINGER EQUATION ON THE EINSTEIN SCHWARZSCHILD MANIFOLD AND FLRW MANIFOLD

RIADH JELLOUL, Carthage Institute of Advanced Commercial Studies,
University of Carthage, Tunis, Tunisia.
ORCID: <https://orcid.org/0009-0002-5085-7400>

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ABSTRACT. In this paper we give different techniques for solving the Schrödinger equation as well as a solution radial developable in Frobenius series on the Einstein Schwarzschild manifold and FLRW manifold.

INTRODUCTION

The Schrodinger equation is the fundamental equation of quantum physics, just as Newton's law is in classical physics. It is used to describe a wide variety of phenomena, whether in quantum optics (laser beam propagation), atomic physics (superconductivity, Bose-Einstein condensation), electronic technology (semiconductors, transistors, memories), plasma physics, astrophysics, electron microscopy, neutronics, chemistry, biology, etc. The equation is written as follows: We denote:

$$\begin{cases} E = h\nu \\ p = \frac{h}{\lambda} \end{cases}$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V\Psi$$

In both expressions above, the letter h denotes Planck's constant and $\hbar = \frac{h}{2\pi}$. We consider the standard Schwarzschild metric of mass M :

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$

and we restrict ourselves to the spatial slice $t = \text{const}$, which is a Riemannian manifold (Σ, g) with metric

$$ds_\Sigma^2 = \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad r > 2M.$$

We denote

$$f(r) = 1 - \frac{2M}{r}.$$

The stationary Schrödinger equation on (Σ, g) is written:

$$-\frac{\hbar^2}{2m} \Delta_g \Psi + V(r)\Psi = E\Psi.$$

1. RADIAL SCHRÖDINGER EQUATION ON THE EINSTEIN SCHWARZSCHILD MANIFOLD

Separation of Variables. By writing

$$\Psi(r, \theta, \varphi) = R(r)Y_{lm}(\theta, \varphi),$$

we obtain for the radial Laplacian:

$$\Delta_g \Psi = \left[\frac{\sqrt{f}}{r^2} \frac{d}{dr} \left(r^2 \sqrt{f} R'(r) \right) - \frac{l(l+1)}{r^2} R(r) \right] Y_{lm}.$$

We thus obtain the radial equation:

$$(1) \quad -\frac{\hbar^2}{2m} \left[f(r) R''(r) + \left(\frac{2}{r} - \frac{3M}{r^2} \right) R'(r) - \frac{l(l+1)}{r^2} R(r) \right] + V(r) R(r) = E R(r).$$

Equation for the reduced function $u(r) = rR(r)$. We set

$$u(r) = rR(r).$$

Equation (1) then becomes

$$(2) \quad -\frac{\hbar^2}{2m} \left[f(r) u''(r) + \frac{M}{r^2} u'(r) - \frac{M + l(l+1)r}{r^3} u(r) \right] + V(r) u(r) = E u(r).$$

A standard form $u'' + P(r)u' + Q(r)u = 0$ is obtained by isolating u'' :

$$(3) \quad u''(r) + P(r)u'(r) + Q(r)u(r) = 0,$$

with

$$P(r) = \frac{M}{r(r-2M)}, \quad Q(r) = -\frac{M + l(l+1)r}{r^3 f(r)} + \frac{2m}{\hbar^2 f(r)} (V(r) - E).$$

Local analysis near the horizon $r = 2M$. We set

$$x = r - 2M, \quad x \rightarrow 0^+.$$

SO

$$f(r) = \frac{x}{2M} + \mathcal{O}(x^2), \quad P(r) = \frac{1}{2x} - \frac{1}{4M} + \mathcal{O}(x),$$

And

$$Q(r) = \frac{q_{-1}}{x} + q_0 + \mathcal{O}(x),$$

with

$$(4) \quad q_{-1} = -\frac{1 + 2l(l+1)}{4M} + \frac{4mM}{\hbar^2} (V(2M) - E).$$

Frobenius Method. We seek a solution in the form:

$$u(x) = x^s \sum_{n=0}^{\infty} c_n x^n, \quad c_0 \neq 0.$$

Substituting this into equation (3), the index equation is:

$$s(s-1) + \frac{s}{2} = 0 \implies s\left(s - \frac{1}{2}\right) = 0.$$

So two exponents:

$$\boxed{s_1 = 0, \quad s_2 = \frac{1}{2}.$$

Series near the horizon.

Regular solution.

$$(5) \quad u(x) = c_0 \left(1 - 2q_{-1}x + \mathcal{O}(x^2) \right), \quad s = 0.$$

Second solution (zeroes at the horizon).

$$(6) \quad u(x) = x^{1/2} \left(c_0 + c_1 x + \mathcal{O}(x^2) \right), \quad s = \frac{1}{2}.$$

Physical Interpretation. Since $R(r) = u(r)/r$:

- $s = 0$: $R(r)$ remains finite and nonzero at the horizon $\Rightarrow$ regular solution.
- $s = 1/2$: $R(r) \sim (r - 2M)^{1/2} \Rightarrow$ the wave function vanishes at the horizon.

The physically relevant solution depends on the imposed boundary conditions (regularity, absorption, radiative conditions in turtle coordinates, etc.).

2. LOCAL SERIES EXPANSION OF THE REGULAR SOLUTION NEAR $r = 2M$

We return to the local variable

$$x := r - 2M, \quad x \rightarrow 0^+.$$

The reduced function $u(r) = rR(r)$ satisfies the ordinary equation

$$u''(r) + P(r)u'(r) + Q(r)u(r) = 0,$$

with

$$P(r) = \frac{M}{r(r - 2M)}, \quad Q(r) = -\frac{M + \ell(\ell + 1)r}{r^3 f(r)} + \frac{2m}{\hbar^2 f(r)}(V(r) - E), \quad f(r) = 1 - \frac{2M}{r}.$$

Coefficient Series. Let us expand P and Q in series in x around $x = 0$:

$$P(r) = \frac{1}{2x} + p_0 + p_1x + p_2x^2 + \mathcal{O}(x^3),$$

$$Q(r) = \frac{q_{-1}}{x} + q_0 + q_1x + q_2x^2 + \mathcal{O}(x^3).$$

The coefficients p_i are directly obtained by expanding P :

$$p_0 = -\frac{1}{4M}, \quad p_1 = \frac{1}{8M^2}, \quad p_2 = -\frac{1}{16M^3}.$$

If we write the expansion of V around $r = 2M$ as

$$V(r) = V_0 + V_1x + \frac{1}{2}V_2x^2 + \frac{1}{6}V_3x^3 + \dots,$$

then we obtain (after expanding Q) the following expressions for the first coefficients q_j :

$$q_{-1} = \frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2}$$

$$q_0 = \frac{8M^2m(-E + 2MV_1 + V_0) + \hbar^2(\ell^2 + \ell + 1)}{4M^2\hbar^2}$$

$$q_1 = \frac{32M^3m(MV_2 + V_1) - \hbar^2(2\ell^2 + 2\ell + 3)}{16M^3\hbar^2}$$

$$q_2 = \frac{16M^4m(2MV_3 + 3V_2) + 3\hbar^2(\ell^2 + \ell + 2)}{48M^4\hbar^2}.$$

Frobenius expansion — regular solution ($s = 0$). We seek the regular solution near $x = 0$ in the form

$$u(x) = \sum_{n=0}^{\infty} c_n x^n, \quad c_0 \neq 0.$$

By identifying the coefficients of the equation series, we obtain the recurrence relations and, for the first coefficients,

$$(\text{order } x^{-1}) : \quad c_0 q_{-1} + \frac{1}{2} c_1 = 0 \quad \Rightarrow \quad \boxed{c_1 = -2 q_{-1} c_0},$$

$$(\text{order } x^0) : \quad c_0 q_0 + c_1 q_{-1} + 3c_2 - \frac{c_1}{4M} = 0 \quad \Rightarrow \quad \boxed{c_2 = \frac{2M(-q_0 + 2q_{-1}^2) - q_{-1}}{6M} c_0},$$

and (continuing the same elimination)

$$\boxed{c_3 = -\frac{1}{90M^2} \left[M^2(24p_1 q_{-1} + 28q_0 q_{-1} - 12q_1 - 8q_{-1}^3) - 2M(q_0 - 3q_{-1}^2) - q_{-1} \right] c_0},$$

and finally (order required for the term in x^4) we obtain the explicit formula ^(*)

$$\boxed{c_4 = -\frac{1}{14} \left(c_0 q_2 + c_1 p_2 + c_1 q_1 + 2c_2 p_1 + c_2 q_0 + c_3 q_{-1} - \frac{3}{4M} c_3 \right)}.$$

The above relations are sufficient to construct the expansion up to order x^4 . Dividing by c_0 defines the normalized coefficients

$$a_n := \frac{c_n}{c_0}, \quad n = 1, 2, 3, 4,$$

which gives:

$$\boxed{a_1 = -2q_{-1}}$$

$$\boxed{a_2 = \frac{2M(-q_0 + 2q_{-1}^2) - q_{-1}}{6M}}$$

$$\boxed{a_3 = -\frac{1}{90M^2} \left[M^2(24p_1 q_{-1} + 28q_0 q_{-1} - 12q_1 - 8q_{-1}^3) - 2M(q_0 - 3q_{-1}^2) - q_{-1} \right]}$$

$$\boxed{a_4 = -\frac{1}{14} \left(q_2 + a_1 p_2 + a_1 q_1 + 2a_2 p_1 + a_2 q_0 + a_3 q_{-1} - \frac{3}{4M} a_3 \right)}.$$

Final Development. Thus, the regular solution around $r = 2M$ is written (for a constant c_0)

$$\boxed{u(x) = c_0 \left[1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \mathcal{O}(x^5) \right]}$$

with a_1, a_2, a_3, a_4 given above and the p_i, q_j specified in terms of $M, \ell, m, \hbar$ and the derivatives of V in $r = 2M$.

Remarks and explicit forms.

- We have already explained above the explicit expressions of q_{-1}, q_0, q_1, q_2 in terms of V_0, V_1, V_2, V_3 . Substituting these quantities into the formulas for a_k gives expressions only in $M, \ell, m, \hbar, E$ and the derivatives $V^{(k)}(2M)$.

- For p_1, p_2 we will use the numerical values:

$$p_1 = \frac{1}{8M^2}, \quad p_2 = -\frac{1}{16M^3}.$$

- The chosen solution corresponds to the exponent $s = 0$ (regular solution, $u(0) \neq 0$, so $R(2M) = u(0)/(2M)$ is finite). The other solution has the exponent $s = \frac{1}{2}$ and includes half-integer powers.

(*) the formula for c_4 is written directly based on the already calculated coefficients c_0, c_1, c_2, c_3 — it can be used immediately to obtain the term in x^4 (replace c_1, c_2, c_3 with the expressions given above).

EXPLICIT EXPRESSIONS OF THE COEFFICIENTS $a_1, \dots, a_4$

We recall the intermediate coefficients (expansions in $x = r - 2M$):

$$\begin{aligned} p_0 &= -\frac{1}{4M}, & p_1 &= \frac{1}{8M^2}, & p_2 &= -\frac{1}{16M^3}, \\ q_{-1} &= \frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2}, \\ q_0 &= \frac{8M^2m(-E + 2MV_1 + V_0) + \hbar^2(\ell^2 + \ell + 1)}{4M^2\hbar^2}, \\ q_1 &= \frac{32M^3m(MV_2 + V_1) - \hbar^2(2\ell^2 + 2\ell + 3)}{16M^3\hbar^2}, \\ q_2 &= \frac{16M^4m(2MV_3 + 3V_2) + 3\hbar^2(\ell^2 + \ell + 2)}{48M^4\hbar^2}. \end{aligned}$$

The normalized coefficients $a_n = c_n/c_0$ are then expressed:

$$a_1 = -2q_{-1} = \frac{-16M^2m(V_0 - E) + \hbar^2(2\ell^2 + 2\ell + 1)}{2M\hbar^2}.$$

$$a_2 = \frac{2M(-q_0 + 2q_{-1}^2) - q_{-1}}{6M}.$$

Replacing q_{-1} and q_0 explicitly gives

$$\begin{aligned} a_2 &= \frac{1}{6M} \left\{ 2M \left[-\frac{8M^2m(-E + 2MV_1 + V_0) + \hbar^2(\ell^2 + \ell + 1)}{4M^2\hbar^2} \right. \right. \\ &\quad \left. \left. + 2 \left(\frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2} \right)^2 \right] \right. \\ &\quad \left. - \frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2} \right\}, \end{aligned}$$

which can be simplified algebraically if desired.

The coefficient a_3 (given previously by a condensed formula) is:

$$a_3 = -\frac{1}{90M^2} \left[M^2(24p_1q_{-1} + 28q_0q_{-1} - 12q_1 - 8q_{-1}^3) - 2M(q_0 - 3q_{-1}^2) - q_{-1} \right].$$

Substituting p_1, q_{-1}, q_0, q_1 we explicitly obtain

$$a_3 = -\frac{1}{90M^2} \left\{ M^2 \left[24 \left(\frac{1}{8M^2} \right) \left(\frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2} \right) \right. \right. \\ + 28 \left(\frac{8M^2m(-E + 2MV_1 + V_0) + \hbar^2(\ell^2 + \ell + 1)}{4M^2\hbar^2} \right) \left(\frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2} \right) \\ - 12 \left(\frac{32M^3m(MV_2 + V_1) - \hbar^2(2\ell^2 + 2\ell + 3)}{16M^3\hbar^2} \right) - 8 \left(\frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2} \right)^3 \Big] \\ - 2M \left(\frac{8M^2m(-E + 2MV_1 + V_0) + \hbar^2(\ell^2 + \ell + 1)}{4M^2\hbar^2} - 3 \left(\frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2} \right)^2 \right) \\ \left. - \frac{16M^2m(V_0 - E) - \hbar^2(2\ell^2 + 2\ell + 1)}{4M\hbar^2} \right\}.$$

Finally, for a_4 we use the compact relation

$$a_4 = -\frac{1}{14} \left(q_2 + a_1p_2 + a_1q_1 + 2a_2p_1 + a_2q_0 + a_3q_{-1} - \frac{3}{4M}a_3 \right).$$

Replacing in this expression $p_1, p_2, a_1, a_2, a_3, q_0, q_1, q_2, q_{-1}$ with the formulas given above, we obtain a_4 entirely as a function of $M, \ell, m, \hbar, E, V_0, V_1, V_2, V_3$. For readability, we can write:

$$a_4 = -\frac{1}{14} \left\{ q_2 + \left(\frac{-16M^2m(V_0 - E) + \hbar^2(2\ell^2 + 2\ell + 1)}{2M\hbar^2} \right) \left(-\frac{1}{16M^3} \right) \right. \\ \left. + a_1q_1 + 2a_2 \left(\frac{1}{8M^2} \right) + a_2q_0 + a_3q_{-1} - \frac{3}{4M}a_3 \right\},$$

with a_1, a_2, a_3 given previously and q_2 equal to

$$q_2 = \frac{16M^4m(2MV_3 + 3V_2) + 3\hbar^2(\ell^2 + \ell + 2)}{48M^4\hbar^2}.$$

3. RADIAL SOLUTION OF THE SCHRÖDINGER EQUATION ON AN FLRW MANIFOLD

Consider the FLRW metric in comoving coordinates $(t, \chi, \theta, \varphi)$:

$$ds^2 = -dt^2 + a(t)^2(d\chi^2 + S_k(\chi)^2d\Omega^2), \quad S_k(\chi) = \begin{cases} \sin \chi, & k = +1, \\ \chi, & k = 0, \\ \sinh \chi, & k = -1. \end{cases}$$

The non-relativistic Schrödinger equation on this manifold is written as

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta_g \Psi + V(t, \chi) \Psi,$$

where Δ_g is the Laplace-Beltrami associated with the spatial metric $a(t)^2(d\chi^2 + S_k(\chi)^2d\Omega^2)$. Since $\Delta_g = \frac{1}{a(t)^2} \Delta_h$, we obtain

$$i\hbar \partial_t \Psi = -\frac{\hbar^2}{2m a(t)^2} \Delta_h \Psi + V(t, \chi) \Psi.$$

We seek spherically symmetric solutions $\Psi(t, \chi)$. The radial Laplacian on the spatial section is equal to

$$\Delta_h f(\chi) = \frac{1}{S_k(\chi)^2} \partial_\chi (S_k(\chi)^2 \partial_\chi f(\chi)).$$

Spatial Separation (Radial Mode). We set the spatial eigenproblem

$$(*) \quad \frac{1}{S_k(\chi)^2} \partial_\chi (S_k(\chi)^2 \partial_\chi R(\chi)) + \lambda R(\chi) = 0.$$

Setting $u(\chi) = S_k(\chi)R(\chi)$ we obtain

$$u''(\chi) + \lambda u(\chi) = 0.$$

Therefore

$$u(\chi) = A \cos(\sqrt{\lambda} \chi) + B \sin(\sqrt{\lambda} \chi),$$

and consequently

$$R(\chi) = \frac{1}{S_k(\chi)} \left[A \cos(\sqrt{\lambda} \chi) + B \sin(\sqrt{\lambda} \chi) \right], \quad \lambda \geq 0.$$

Temporal Separation (case $V = 0$). We seek $\Psi(t, \chi) = A(t) R(\chi)$. We obtain

$$i\hbar \frac{A'(t)}{A(t)} = \frac{\hbar^2 \lambda}{2m a(t)^2},$$

so

$$A(t) = A_0 \exp \left(-i \frac{\hbar \lambda}{2m} \int^t \frac{dt'}{a(t')^2} \right).$$

In order to preserve probabilistic normalization, we choose the factor $a(t)^{-3/2}$:

$$\Psi(t, \chi) = \mathcal{N} a(t)^{-3/2} R(\chi) \exp \left(-i \frac{\hbar \lambda}{2m} \int^t \frac{dt'}{a(t')^2} \right).$$

Final radial solution. We obtain the family of radial solutions ($l = 0$):

$$\Psi_\lambda(t, \chi) = \mathcal{N} a(t)^{-3/2} \frac{A \cos(\sqrt{\lambda} \chi) + B \sin(\sqrt{\lambda} \chi)}{S_k(\chi)} \exp \left(-i \frac{\hbar \lambda}{2m} \int^t \frac{dt'}{a(t')^2} \right).$$

Flat case $k = 0$. Since $S_0(\chi) = \chi$, the regular solution at $\chi = 0$ ($A = 0$) gives:

$$\Psi(t, r) = \mathcal{N} a(t)^{-3/2} \frac{\sin(\kappa r)}{r} \exp \left(-i \frac{\hbar \kappa^2}{2m} \int^t \frac{dt'}{a(t')^2} \right), \quad \kappa = \sqrt{\lambda}.$$

These solutions form a basis for the radial modes (generalized spherical harmonics $l = 0$) on the spatial section of the FLRW metric.

EXPLICIT EXAMPLES: RADIAL SOLUTIONS ON FLRW FOR $k = +1$ AND $k = -1$

We consider the FLRW metric in comoving coordinates $(t, \chi, \theta, \varphi)$:

$$ds^2 = -dt^2 + a(t)^2 (d\chi^2 + S_k(\chi)^2 d\Omega^2), \quad S_k(\chi) = \begin{cases} \sin \chi, & k = +1, \\ \sinh \chi, & k = -1. \end{cases}$$

The non-relativistic Schrödinger equation (zero potential $V = 0$) is

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m a(t)^2} \Delta_h \Psi,$$

where for a radial function $R(\chi)$, the spatial Laplacian is equal to

$$\Delta_h R(\chi) = \frac{1}{S_k(\chi)^2} \frac{d}{d\chi} \left(S_k(\chi)^2 \frac{dR}{d\chi} \right).$$

1) **Case $k = +1$ (sphere S^3).** Let $u(\chi) = \sin \chi R(\chi)$. The eigenequation $\Delta_h R + \lambda R = 0$ becomes

$$u''(\chi) + (\lambda + 1)u(\chi) = 0.$$

The regular solutions on $[0, \pi]$ are

$$u_n(\chi) = \sin(n\chi), \quad n = 1, 2, 3, \dots$$

and the corresponding radial functions are

$$R_n(\chi) = \frac{\sin(n\chi)}{\sin \chi}, \quad \lambda_n = n^2 - 1.$$

The time solution for each mode n is

$$\Psi_n(t, \chi) = \mathcal{N}_n a(t)^{-3/2} \frac{\sin(n\chi)}{\sin \chi} \exp\left(-i \frac{\hbar \lambda_n}{2m} \int^t \frac{dt'}{a(t')^2}\right).$$

2) **Case $k = -1$ (hyperbolic space H^3).** Let $u(\chi) = \sinh \chi R(\chi)$. The eigenequation becomes

$$u''(\chi) + (\lambda - 1)u(\chi) = 0.$$

For a continuous spectrum $\lambda > 1$, the regular solutions at $\chi = 0$ are

$$R_\mu(\chi) = \frac{\sin(\mu\chi)}{\sinh \chi}, \quad \lambda = \mu^2 + 1, \quad \mu \geq 0.$$

The complete time solution is

$$\Psi_\mu(t, \chi) = \mathcal{N}(\mu) a(t)^{-3/2} \frac{\sin(\mu\chi)}{\sinh \chi} \exp\left(-i \frac{\hbar \lambda}{2m} \int^t \frac{dt'}{a(t')^2}\right).$$

These solutions are regular at $\chi = 0$ and represent the analogue of the spherical functions $\frac{\sin(\kappa r)}{r}$ of the flat case $k = 0$.

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